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Black holes, elementary strings and holomorphic anomaly

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Abstract: In a previous paper we had proposed a specific route to relating the entropy of two charge black holes to the degeneracy of elementary string states in N=4 supersymmetric heterotic string theory in four dimensions. For toroidal compactification this proposal works correctly to all orders in a power series expansion in inverse charges provided we take into account the corrections to the black hole entropy formula due to holomorphic anomaly. In this paper we demonstrate that similar agreement holds also for other N=4 supersymmetric heterotic string compactifications.

Keywords: Superstrings and Heterotic Strings, Black Holes in String Theory.

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Contents

1. Introduction

The attempt to relate the Bekenstein-Hawking entropy of a black hole to the number of states of the black hole in some microscopic description of the theory is quite old. In string theory this takes a new direction as the theory already has a large number of massive states in the spectrum of elementary string, and hence one is tempted to speculate that for large mass we should be able to relate the degeneracy of these states to the entropy of a black hole with the same charge quantum numbers [1−5]. However this problem is complicated due to large renormalization effects which make it difficult to relate the quantum numbers of the black hole to those of the elementary string states. This problem can be avoided by considering a tower of BPS states [6−8] where such renormalization effects are absent. The entropy of the corresponding black hole solution vanishes in the supergravity approximation; however one finds that the curvature associated with the solution becomes large near the horizon, indicating breakdown of the supergravity approximation. Although the complete analysis of the problem has not been possible to this date, a general argument based on symmetries of the theory shows that the entropy of the black hole, modified by α' corrections, has the right dependence on all the parameters (charges as well as the asymptotic vacuum expectation values of various fields) so as to agree with the logarithm of the degeneracy of elementary string states [9−11]. However the overall normalization constant is not determined by the symmetry argument, and its computation requires inclusion of all order α' corrections to the tree level heterotic string action.

It was later realized that instead of elementary strings, D-branes [12] provide a much richer arena for the study of black holes. In particular, by considering a sufficiently complicated configuration of D-branes one can ensure that the corresponding BPS black hole solution carrying the same charge quantum numbers as the D-brane system has a finite area event horizon where α' and string loop corrections are small. Comparison of the entropy of the black hole to the logarithm of the degeneracy of states of the D-brane configuration (which we shall call the statistical entropy) shows exact agreement [13] for large charges. This agreement has been verified for a variety of black holes in different string theories.

Initial comparison between the black hole entropy and the statistical entropy was carried out in the limit of large charges. For a class of black holes in $N = 2$ supersymmetric string compactification [14 – 16] ref. [17] attempted to go beyond the large charge limit, and computed corrections to the statistical entropy which are suppressed by the inverse power of the charges. The corresponding corrections to the black hole entropy come from higher derivative terms in the effective action. By taking into account a special class of higher derivative terms [18, 19] which come from supersymmetrization of the curvature squared terms in the action [20, 21], refs. [22 – 28] computed corrections to the black hole entropy and found precise agreement. One non-trivial aspect of this calculation is that in order to reach this agreement we need to also modify the Bekenstein-Hawking formula for the black hole entropy due to the presence of higher derivative terms [29 – 32].

Recently there has been renewed interest in the black hole solution representing elementary string states. This followed the observation by Dabholkar [33] that if we take into account the special class of higher derivative terms which were used in the analysis of [22 – 28] and calculate their effect on the black hole solutions representing elementary string states, we get a solution with a finite area event horizon. The entropy of this black hole, calculated using the formulæ given in [29 – 32], reproduces precisely the leading term in the expression for the statistical entropy obtained by taking the logarithm of the degeneracy of elementary string states. This analysis was developed further in [34 – 37]. An alternative viewpoint for these black holes can be found in [38, 39].

One of the advantages of using elementary string states for comparison with black hole entropy is that for this system the degeneracy of states and hence the statistical entropy is known very precisely. Hence one can try to push this comparison beyond the large charge approximation. However one problem that one encounters in this process is that even if we know the degeneracies exactly, the definition of the statistical entropy is somewhat ambiguous since it depends on the particular ensemble we use to define entropy. As in the case of an ordinary thermodynamic system, all definitions of entropy agree when the charge (analog of volume) is large, but finite 'size' corrections to the entropy differ between the entropies defined through different ensemble. This is due to the fact that the agreement between different ensembles (e.g. microcanonical, canonical and grand canonical ensembles) is proved using a saddle point approximation which is valid only in the 'large volume' limit. Thus the question that we need to address is: which definition of statistical entropy should we use for comparison with the black hole entropy?

There is no *a priori* answer to this question, and one has to proceed by trial and error to discover if there is some natural definition of statistical entropy which agrees with the black hole entropy beyond leading order. For a class of black holes in $N = 2$ supersymmetric string compactification, refs. [40, 41] proposed such a definition based on a mixed ensemble where we sum over half of the charges (the 'electric' charges) by introducing chemical potentials for these charges and keep the other half of the charges fixed. By applying the same prescription to the black holes representing elementary string states in $N = 4$ supersymmetric theories, [33] was able to reproduce the black hole entropy to all orders in the inverse charges up to an additive logarithmic piece which appears as a multiplicative factor in the partition function involving powers of the winding number charge [42]. One disadvantage of this prescription is that it destroys manifest symmetry between the momentum and winding charges of the string since in defining the ensemble we sum over all momentum states but keep fixed the winding charge. As a result T-duality invariance of the statistical entropy defined this way is not guaranteed.

A related but somewhat different proposal for relating the degeneracy of elementary string states to black hole entropy, which maintains manifest T-duality invariance, was proposed in [35]. This also requires summing over charges but in a manner that preserves manifest T-duality. In particular the chemical potential couples to a T-duality invariant combination of the charges.[1] It was shown in [35] that up to terms which are non-perturbative in the inverse charges, this definition of the statistical entropy agrees with the black hole entropy including logarithmic terms, provided we take into account the effect of holomorphic anomaly [43, 44] in the effective action. A related duality invariant presecription for dealing with 1/4 BPS black holes in N=4 supersymmetric heterotic string compactification was later given in [45].

In order to put this proposal on a firm footing it is important that we test it for other N=4 heterotic string compactifications. This is what we attempt to do in this paper. In particular we focus on a class of four dimensional CHL models with $N = 4$ supersymmetry $[46-51]$ and compare the statistical entropy computed using the prescription of [35] with the black hole entropy. We again find that after taking into account corrections due to holomorphic anomaly, the black hole entropy and the statistical entropy agree up to non-perturbative terms.[2]

The rest of the paper is organised as follows. In section 2 we review the proposal of [35] for relating the black hole entropy to the degeneracy of elementary string states. In section 3 we review CHL string compactifications, count the degeneracy of elementary string states in these models, and compute the statistical entropy using these results. In section 4 we

[1] We also sum over all angular momentum states which is equivalent to choosing an ensemble with a chemical potential coupled to the angular momentum, and then extremizing the corresponding free energy with respect to this chemical potential. This sets the chemical potential to zero. This argument is due to B. Pioline, and I wish to thank A. Dabholkar for discussion on this point.

[2] In the analysis of this paper as well as in the analysis of [35] an overall charge independent additive constant in the expression for the entropy could not be fixed due to our lack of precise knowledge of the effect of the holomorphic anomaly terms on black hole entropy. Thus we could not compare this overall additive constant between the black hole and the statistical entropy.

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calculate the entropy of the black holes of the CHL model carrying the same charge quantum numbers as the elementary string states, and show that the result agrees with the statistical entropy found in section 3. During this computation we also determine the coefficient of the holomorphic anomaly term in these CHL models. Section 5 contains a discussion of the results and possible extension to more general class of models and/or states. The two appendices are devoted to the analysis of the errors involved in the various approximations used in this paper, and to demonstrate that these corrections are all non-perturbative in the inverse charges. Of the two appendices, appendix A analyses the possible errors in the computation of the statistical entropy and appendix B examines possible errors in the computation of the black hole entropy. In appendix B we also determine the S-duality invariant form of the curvature squared terms in the CHL models.

I have been informed by A. Dabholkar that for a general class of models, ref. [52] has successfully carried out the comparison between the black hole entropy and the entropy defined through the ensemble introduced in [40]. After completing this paper we also learned of ref. [53] where some of the computations of section 4 and appendix B, required for determining the form of the curvature squared terms in the effective action, have been carried out.

2. Proposal for relating black hole entropy to the degeneracy of elementary string states

We shall be considering $N = 4$ supersymmetric heterotic string theory in four dimension, with a compactification manifold of the form $K_5 \times S^1$. In this theory we consider a fundamental string wound w times along the circle S^1 and carrying n units of momentum along the same circle. Let $d(n, w)$ denote the degeneracy of elementary string states satisfying the following properties:

- The state is invariant under half of the space-time supersymmetry transformations.
- The state carries gauge charges appropriate to an elementary heterotic string carrying w units of winding and n units of momentum along S^1. This means that if x^4 denotes the coordinate along S^1 and x^μ $(0 \le \mu \le 3)$ denote the coordinates of the non-compact part of the space-time, then the state carries gauge charges proportional to n and w associated with the gauge fields $G^{(10)}_{4\mu}$ and $B^{(10)}_{4\mu}$ respectively, but does not carry any other gauge charge. Here $G^{(10)}_{MN}$ and $B^{(10)}_{MN}$ denote the ten dimensional string metric and the anti-symmetric tensor field respectively.

We shall see in section 3 that the degeneracy of such states is a function of the combination $N \equiv nw$. Denoting this by d_N, we define the partition function associated with these states as:

$$e^{\mathcal{F}(\mu)} = \sum_N d_N \, e^{-\mu N} \, . \tag{2.1}$$

Given $\mathcal{F}(\mu)$, we define the statistical entropy $\widetilde{S}_{\rm stat}$ as the Legendre transform of $\mathcal{F}(\mu)$:

$$\widetilde{S}_{\rm stat}(N) = \mathcal{F}(\mu) + \mu \, N \, , \tag{2.2}$$

where μ is given by the solution of the equation

$$\frac{\partial \mathcal{F}}{\partial \mu} + N = 0\,. \tag{2.3}$$

The proposal of ref. [35] is to identify $\widetilde{S}_{\rm stat}(nw)$ with the entropy of the black hole solution carrying same charge quantum numbers (n, w):

$$\widetilde{S}_{\rm stat}(nw) = S_{BH}(n, w)\,. \tag{2.4}$$

This is the relation we shall try to verify in this paper for different heterotic string compactifications.

The definition of statistical entropy given above is appropriate for a kind of grand canonical ensemble where we introduce a chemical potential conjugate to nw. A more direct definition of the statistical entropy would be the one based on the microcanonical ensemble:

$$S_{\rm stat}(N) = \ln d_N\,. \tag{2.5}$$

The two definitions agree in the limit of large N where we can evaluate the sum in (2.1) by a saddle point method. In this case the leading contribution to $e^{\mathcal{F}(\mu)}$ is given by $d_{N_0} e^{-\mu N_0}$ where N_0 is the value of N that maximizes the summand:

$$\mathcal{F}(\mu) \simeq \ln d_{N_0} - \mu N_0\,, \qquad \frac{\partial}{\partial N_0} \ln d_{N_0} - \mu = 0\,. \tag{2.6}$$

Thus in this approximation $\mathcal{F}(\mu)$ is the Legendre transform of $\ln d_{N_0} = S_{\rm stat}(N_0)$. Hence $\widetilde{S}_{\rm stat}(N)$, defined as the Legendre transform of $\mathcal{F}(\mu)$, will be equal to $S_{\rm stat}(N)$. However the complete $\mathcal{F}(\mu)$ defined through (2.1) has additional contribution besides that given in (2.6), and as a result $S_{\rm stat}$ and $\widetilde{S}_{\rm stat}$ differ in non-leading terms. In particular the coefficient of the $\ln N$ terms in $S_{\rm stat}$ and $\widetilde{S}_{\rm stat}$ differ. It is not *a priori* clear which definition of statistical entropy we should be comparing with the entropy of the black hole solution carrying the same quantum numbers. It was shown in [35] that for heterotic string theory compactified on a torus, $\widetilde{S}_{\rm stat}$ agrees with the black hole entropy up to exponentially suppressed contributions. We shall see in section 4 that such agreement between $\widetilde{S}_{\rm stat}$ and S_{BH} continues to hold also for CHL compactification [46, 47, 50, 51] of the heterotic string theory.

Note that given $S_{\rm stat} = \ln d_N$ we can calculate $\widetilde{S}_{\rm stat}$ using eqs. (2.1)–(2.3). Conversely, given $\widetilde{S}_{\rm stat}$ we can compute $\mathcal{F}(\mu)$ by taking its Legendre transform and then compute d_N by solving (2.1). This gives $S_{\rm stat}$. Thus $S_{\rm stat}$ and $\widetilde{S}_{\rm stat}$ contain complete information about each other and the degeneracies d_N. This allows us to restate the proposal (2.4) in a slightly different but equivalent form. Given $S_{BH}(n, w)$ (which turns out to be a function of the combination $N = nw$) we define $\mathcal{F}_{BH}(\mu)$ by taking the Legendre transform of S_{BH} with respect to the variable N, and then define d_N^{BH} through an analog of eq.(2.1) with $\mathcal{F}(\mu)$ and d_N replaced by $\mathcal{F}_{BH}(\mu)$ and d_N^{BH} respectively. The proposal (2.4) then translates to the relation:

$$\mathcal{F}_{BH}(\mu) = \mathcal{F}(\mu)\,, \qquad d_N^{BH} = d_N\,. \tag{2.7}$$

Although we shall work with (2.4) for convenience, we should keep in mind that verifying (2.4) amounts to verifying (2.7).

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3. Counting degeneracy of BPS string states in CHL models

In this section we shall compute d_N and hence $\mathcal{F}(\mu)$ for a class of $N=4$ supersymmetric heterotic string compactification. First we shall illustrate the counting procedure in the context of a specific CHL model [47] and then generalize this to other models. The construction of the model begins with $E_8 \times E_8$ heterotic string theory compactified on a six torus $T^4 \times \widetilde{S}^1 \times S^1$. In this theory the gauge fields in the Cartan subalgebra consist of 22 gauge fields arising out of the left-moving U(1) currents of the world-sheet theory, and six gauge fields arising out of the right-moving U(1) currents of the world-sheet theory. All the gauge fields associated with the $E_8 \times E_8$ group arise out of the left-moving world-sheet currents. We now mod out this theory by a $\mathbb{Z}_2$ transformation that involves a half shift along $\widetilde{S}^1$ together with an exchange of the two E_8 lattices [47]. The resulting theory still has $N=4$ supersymmetry. In particular the 6 U(1) gauge fields associated with the right-moving world-sheet currents are untouched by the $\mathbb{Z}_2$ projection, and continue to provide us with the graviphoton fields of $N=4$ supergravity. On the other hand only the diagonal sum of the two E_8 gauge groups survive the projection. As a result the $E_8 \times E_8$ component of the gauge group is reduced to E_8, and the rank of the unbroken gauge group from the left-moving sector of the world-sheet is reduced to 14 from its original value 22. 8 of these U(1) gauge fields come from the surviving E_8 gauge group and the other 6 come from appropriate linear combination of the metric and antisymmetric tensor field, with one index lying along one of the six directions of the internal torus and the other index lying along one of the non-compact directions.

We now consider an elementary string state wound w times along S^1 and carrying n units of momentum along the same S^1. The BPS excitations of this string state come from restricting the right-moving oscillators to have total level 1/2 (in the Neveu-Schwarz sector) or 0 (in the Ramond sector) but allowing arbitrary oscillator and momentum excitations in the left-moving sector. We would like to count BPS states with a given set of gauge charges, notably those carried by an elementary string state with w units of winding and n units of momentum along S^1. First let us do this calculation for heterotic string theory on a torus [6]. In this case the only possible excitations are those created by left-moving oscillators, since any additional momentum and/or winding will generate additional gauge charges carried by the state. If N_L denotes the total level of the left-moving oscillators then the level matching condition gives $N_L = nw + 1$, and hence the degeneracy of elementary string states carrying quantum numbers (n, w) is the number of ways we can get total oscillator level N_L from the 24 left-moving oscillators, multiplied by a factor of 16 that counts the degeneracy of the ground state of the right-moving sector. We shall call this number $d^{(0)}_{N_L}$. It is given by the generating function [6]

$$\sum_{N_L} d^{(0)}_{N_L} e^{-\mu(N_L-1)} = 16 \left(\eta(e^{-\mu})\right)^{-24} , \qquad \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) . \tag{3.1}$$

For the CHL string theory under consideration, the counting is a little more complicated. Since only the diagonal E_8 gauge group survives, we can satisfy the condition for vanishing E_8 charge if we choose equal and opposite momentum vector $\vec{p}$ and $-\vec{p}$ from the two E_8

lattices. We choose the overall normalization of $\vec{p}$ such that the E_8 lattice is self-dual under the inner product $(\vec{p},\vec{q}) = \vec{p}\cdot\vec{q}$. In this normalization there is one lattice point per unit volume in the $\vec{p}$ space, and the contribution to the $\bar{L}_0$ eigenvalue from the vector $\vec{s} = (\vec{p}, -\vec{p})$ is given by $\vec{p}^2/2 + \vec{p}^2/2 = \vec{p}^2$. The level matching condition now gives:

$$N_L + \vec{p}^2 = nw + 1\,. \tag{3.2}$$

Thus the degeneracy d_{nw} for given (n, w) is equal to the number of ways we can satisfy (3.2), subject to the condition that the resulting state is even under the orbifold group:

$$d_{nw} \simeq \frac{1}{2}\sum_{N_L}\sum_{\vec{p}\in\Lambda_{E_8}} d^{(0)}_{N_L}\,\delta_{N_L+\vec{p}^2-nw,1}\,. \tag{3.3}$$

Since we only include states which are even under the $\mathbb{Z}_2$ transformation, we must symmetrize the state under the exchange of the oscillators and momenta associated with the two E_8 factors. As shown in appendix A, up to exponentially small contribution this introduces a multiplicative factor of 1/2 in the counting of states which we have included in the right hand side of (3.3). Note that the twisted sector states do not play any role in this counting, since they carry half-integral winding number along the circle $\widetilde{S}^1$ and hence belong to a different charge sector. Using (3.3) and (3.1) the partition function $e^{\mathcal{F}(\mu)}$ defined in (2.1) is now given by

$$e^{\mathcal{F}(\mu)} \simeq \frac{1}{2}\sum_{N_L} d^{(0)}_{N_L} e^{-\mu(N_L-1)} \sum_{\vec{p}\in\Lambda_{E_8}} e^{-\mu\vec{p}^2} = 8\,\left(\eta(e^{-\mu})\right)^{-24} \sum_{\vec{p}\in\Lambda_{E_8}} e^{-\mu\vec{p}^2}\,. \tag{3.4}$$

We shall be interested in the behaviour of $\mathcal{F}(\mu)$ at small μ. In this limit,

$$\left(\eta(e^{-\mu})\right)^{-24} \simeq e^{4\pi^2/\mu}\,\left(\frac{\mu}{2\pi}\right)^{12}\,, \tag{3.5}$$

and, using Poisson resummation formula,

$$\sum_{\vec{p}\in\Lambda_{E_8}} e^{-\mu\vec{p}^2} = \left(\frac{\pi}{\mu}\right)^4 \sum_{\vec{q}\in\Lambda_{E_8}} e^{-\pi^2\vec{q}^2/\mu} \simeq \left(\frac{\pi}{\mu}\right)^4\,. \tag{3.6}$$

Here we have used the fact that the E_8 lattice is self-dual. Thus we get, for small μ

$$e^{\mathcal{F}(\mu)} \simeq \frac{1}{2}\,\left(\frac{\mu}{2\pi}\right)^8 e^{4\pi^2/\mu}\,, \tag{3.7}$$

and hence

$$\mathcal{F}(\mu) \simeq \frac{4\pi^2}{\mu} + 8\ln\frac{\mu}{2\pi} + \ln\frac{1}{2}\,. \tag{3.8}$$

Before we turn to the more general case, let us try to estimate the error in (3.8). The first source of error appears in (3.4) where we have represented the symmetry requirement of the states under the $\mathbb{Z}_2$ orbifold group by a factor of $\frac{1}{2}$ in $e^{\mathcal{F}}$. A more careful analysis described in appendix A shows that the error in $\mathcal{F}$ due to this approximation involves powers of $e^{-\pi^2/\mu}$. The second source of error is in the small μ approximation of $\eta(e^{-\mu})$ used

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in (3.5). The fractional error in this formula is of order $e^{-4\pi^2/\mu}$. Finally the approximation used in (3.6) also introduces a fractional error involving powers of $e^{-\pi^2/\mu}$. Thus we conclude that the net error in eq. (3.8) for $\mathcal{F}(\mu)$ is non-perturbative in the small μ approximation.

The above analysis can be easily generalized to a class of other CHL compactifications. We begin with heterotic string theory compactified on $T^4 \times \widetilde{S}^1 \times S^1$, and tune the moduli associated with T^4 compactification such that the twentyfour dimensional Narain lattice [54, 55] $\Lambda_{20,4}$ associated with heterotic compactification on T^4 has a $\mathbb{Z}_m$ symmetry. We now mod out the theory by a $\mathbb{Z}_m$ symmetry group generated by a shift h of order m along $\widetilde{S}^1$, accompanied by the generator g of the $\mathbb{Z}_m$ automorphism group of $\Lambda_{20,4}$. In order that the final theory has $N = 4$ world-sheet supersymmetry, the $\mathbb{Z}_m$ automorphism should act trivially on the right-moving U(1) currents of the world-sheet. However it could have non-trivial action on the left-moving world-sheet currents, and as a result modding out by this symmetry projects out certain number (say k) of the U(1) gauge fields belonging to the Cartan subalgebra of the gauge group. In the resulting quotient theory the rank of the gauge group associated with the left-moving sector of the world-sheet theory is reduced to $(22 - k)$. If we now consider an elementary string wound w times along S^1 and carrying n units of momentum along S^1, then the computation of the degeneracy d_N $(N = nw)$ and the partition function $e^{\mathcal{F}(\mu)}$ associated with these states involves a sum over the oscillator levels N_L as well as a sum over the k-dimensional momentum lattice Λ whose vectors do not couple to any massless gauge field of the resulting theory. This gives

$$d_N \simeq \frac{1}{m} \sum_{N_L} \sum_{\vec{s}\in\Lambda} d^{(0)}_{N_L}\, \delta_{N_L+\vec{s}^2/2-N,1}\,, \tag{3.9}$$

and

$$e^{\mathcal{F}(\mu)} \simeq \frac{1}{m} \sum_{N_L} d^{(0)}_{N_L}\, e^{-\mu(N_L-1)} \sum_{\vec{s}\in\Lambda} e^{-\mu\vec{s}^2/2}\,. \tag{3.10}$$

As discussed in appendix A, the factor of $1/m$ approximately accounts for the fact that we need to count only those states which are invariant under the orbifold group, and the error involved in this approximation involves powers of $e^{-\pi^2/\mu}$. The sum over N_L can be performed using (3.1), whereas the sum over $\vec{s}$ can be done using Poisson resummation formula:

$$\sum_{\vec{s}\in\Lambda} e^{-\mu\vec{s}^2/2} = \frac{1}{V}\left(\frac{2\pi}{\mu}\right)^{k/2} \sum_{\vec{r}\in\widetilde{\Lambda}} e^{-2\pi^2\vec{r}^2/\mu} \simeq \frac{1}{V}\left(\frac{2\pi}{\mu}\right)^{k/2}\,, \tag{3.11}$$

where V denotes the volume of the unit cell in the lattice Λ and $\widetilde{\Lambda}$ is the lattice dual to Λ. Thus the final expression for $\mathcal{F}(\mu)$ is given by

$$\mathcal{F}(\mu) \simeq \frac{4\pi^2}{\mu} + \frac{1}{2}\,(24-k)\ln\frac{\mu}{2\pi} + \ln\frac{16}{Vm}\,. \tag{3.12}$$

The errors in this equation come from errors in eqs.(3.5), (3.10) and (3.11). Each of these errors involves powers of $e^{-\pi^2/\mu}$. Thus as in the first example, for small μ the corrections to (3.12) involve powers of $e^{-\pi^2/\mu}$.

Given $\mathcal{F}(\mu)$, we define the statistical entropy $\widetilde{S}_{\rm stat}$ through (2.2), (2.3). This gives:

$$\widetilde{S}_{\rm stat}(N) \simeq \mu\, N + \frac{4\pi^2}{\mu} + \frac{1}{2}\,(24-k)\ln\frac{\mu}{2\pi} + \ln\frac{16}{Vm}\,, \tag{3.13}$$

where μ is the solution of the equation

$$-\frac{4\pi^2}{\mu^2} + \frac{24-k}{2\mu} + N \simeq 0\,, \tag{3.14}$$

and $N = nw$. In the limit of large N, the μ obtained by solving (3.14) is given by

$$\mu \simeq \frac{2\pi}{\sqrt{N}}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right). \tag{3.15}$$

Thus for large N, μ is small. This justifies the small μ approximation used in arriving at (3.12). Since the error in $\mathcal{F}(\mu)$ involves powers of $e^{-\pi^2/\mu}$, the error in $\widetilde{S}_{\rm stat}$ computed from (3.13), (3.14) will involve powers of $e^{-\pi\sqrt{N}}$.

We conclude this section by noting that $\widetilde{S}_{\rm stat}$ computed from (3.13), (3.14) is of the form

$$\widetilde{S}_{\rm stat} = 4\pi\sqrt{N} - \frac{24-k}{2}\,\ln\sqrt{N} + \mathcal{O}(1)\,. \tag{3.16}$$

Although eq. (3.16) gives more explicit expression for $\widetilde{S}_{\rm stat}$, this equation has corrections involving inverse powers of $\sqrt{N}$. Thus the comparison with the black hole entropy will be made with the formulæ (3.13), (3.14) for $\widetilde{S}_{\rm stat}$ which are correct up to error terms involving powers of $e^{-\pi\sqrt{N}}$.

4. Analysis of black hole entropy and comparison with the statistical entropy

We shall now turn to the analysis of the entropy S_{BH} of the BPS black hole carrying the same charge and mass as an elementary string state described above. The entropy of such a black hole vanishes in the supergravity approximation [9]. However the curvature associated with the string metric becomes large near the horizon, showing that we must take the higher derivative terms into account for computing the entropy of such a black hole. In contrast the string coupling near the horizon is small for large n and w and hence to leading order we can ignore the string loop corrections [9]. There is a general symmetry argument that shows that at the tree level in heterotic string theory the modified entropy must have the form $a\sqrt{nw}$ for some numerical constant a [9, 11, 35]. However the value of the constant a is not determined by this argument (a could be zero for example). If $a = 4\pi$, the black hole entropy would agree with the leading term in the expression (3.16) for $\widetilde{S}_{\rm stat}$. Following the formalism developed in refs. [22−27], ref. [33] analyzed the effect of a special class of higher derivative terms in the tree level effective action of heterotic string theory which come from supersymmetric completion of the term [18, 19]

$$\frac{1}{16\pi}\int d^4x\,\sqrt{\det g}\,\left(S\,R^-_{\mu\nu\rho\sigma}\,R^{-\mu\nu\rho\sigma} + \bar{S}\,R^+_{\mu\nu\rho\sigma}\,R^{+\mu\nu\rho\sigma}\right), \tag{4.1}$$

where $g_{\mu\nu}$, $R^{\pm}_{\mu\nu\rho\sigma}$ and S denote respectively the canonical metric, the self-dual and anti-self-dual components of the Riemann tensor and the complex scalar field whose real and imaginary parts are given by the exponential of the dilaton field and the axion field respectively. After taking into account the modification of the equations of motion and supersymmetry transformation laws due to these additional terms, the modified black hole entropy is given by the expression [33–36]:

$$S_{BH} = \frac{\pi N}{S_0} + 4\pi\, S_0\,, \qquad N \equiv nw\,, \tag{4.2}$$

where S_0, defined as the value of the field S at the horizon, is given by the solution of the equation[3]

$$-\frac{\pi\, N}{S_0^2} + 4\pi = 0\,. \tag{4.3}$$

This gives $S_{BH} = 4\pi\sqrt{N}$. This agrees with the leading term in the expression (3.16) for $\widetilde{S}_{\rm stat}$ [33].

Ref. [33] checked this agreement for heterotic string compactification on a torus. However once this has been checked for torus compactification, similar agreement for other heterotic string compactifications is automatic due to an argument in [11] where it was shown that at tree level in heterotic string theory the part of the effective action relevant for computing the entropy of these states is identical in all heterotic string compactification with $N = 4$ or $N = 2$ supersymmetry. Thus the leading contribution to S_{BH} will be given by $4\pi\sqrt{nw}$ for all heterotic string compactifications. This clearly agrees with the leading term in the expression (3.16) for $\widetilde{S}_{\rm stat}$.

We now turn to the non-leading corrections to the entropy. For this we need to go beyond the tree level effective action of the heterotic string theory. A special class of such corrections come from a term in the action of the form:

$$-\frac{K}{128\pi^2}\int d^4x\,\sqrt{\det g}\,\ln(S+\bar{S})\,R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma}\,, \tag{4.4}$$

that arises from the so called holomorphic anomaly [43, 44]. Here K is a constant that will be determined later. (For toroidal compactification $K = 24$ [21].) In order to carry out a systematic analysis of the effect of this term on the expression for the black hole entropy, we need to

- supersymmetrize this term,
- study how these additional terms in the action modify the expression for the black hole entropy in terms of various fields near the horizon,
- study how the various field configurations near the horizon are modified by these extra terms in the equation of motion,

[3] Note that the left hand side of (4.3) is equal to the derivative of the right hand side of (4.2) with respect to S_0. This feature survives even after including the correction due to holomorphic anomaly [35, 45].

- and finally evaluate the modified expression for the black hole entropy for the modified near horizon field configuration.

This however has not so far been carried out explicitly. In order to appreciate the reason for this difficulty, one needs to know the difference in the origin of the terms (4.1) and (4.4). In fact both terms originate from a term of the form:[4]

$$\int d^4x \sqrt{\det g}\, \left[\phi(S,\bar{S})\, R^-_{\mu\nu\rho\sigma} R^{-\mu\nu\rho\sigma} + \text{c.c.}\right], \tag{4.5}$$

where

$$\phi(S,\bar{S}) = g(S) - \frac{K}{128\pi^2}\, \ln(S+\bar{S}) \tag{4.6}$$

is the sum of a piece $g(S)$ that is holomorphic in S and a piece proportional to $\ln(S+\bar{S})$ that is a function of both S and $\bar{S}$. For large S,

$$g(S) \simeq \frac{S}{16\pi}, \tag{4.7}$$

so as to reproduce (4.1). Hence

$$\phi(S,\bar{S}) \simeq \frac{1}{16\pi}\left(S - \frac{K}{8\pi}\ln(S+\bar{S})\right). \tag{4.8}$$

A detailed analysis of the function $g(S)$ can be found in appendix B where it has been shown that corrections to (4.7) are of order $e^{-2\pi S}$. The contribution (4.1) comes from the $g(S) \simeq S/16\pi$ term in $\phi(S,\bar{S})$. Being holomorphic in S, this part is easy to supersymmetrize, and was used in arriving at expressions (4.2), (4.3) for S_{BH}. On the other hand (4.4) arises from the part of $\phi(S,\bar{S})$ proportional to $\ln(S+\bar{S})$ which cannot be regarded as a holomorphic function. Supersymmetrization of this term has not been carried out completely. Nevertheless, using various consistency requirements, [24] guessed that supersymmetric completion of the term (4.5) modifies equations (4.2) and (4.3) to[5]

$$S_{BH} = \frac{\pi N}{S_0} + 64\pi^2\, g(S_0) - \frac{K}{2}\, \ln(2S_0), \tag{4.9}$$

and

$$-\frac{\pi\, N}{S_0^2} + 64\pi^2 g'(S_0) - \frac{K}{2S_0} \simeq 0. \tag{4.10}$$

For large N, S_0 computed from (4.10) is of order $\sqrt{N}$ and hence we can use the large S_0 approximation (4.7) for $g(S_0)$. This gives

$$S_{BH} \simeq \frac{\pi N}{S_0} + 4\pi\, S_0 - \frac{K}{2}\, \ln(2S_0), \tag{4.11}$$

[4]In the convention of ref. [24] this corresponds to choosing $F^{(1)} = -\frac{i\pi}{4}\left(g(S) - \frac{K}{64\pi^2}\ln(S+\bar{S})\right)\Upsilon$. For toroidal compactification $g(S) = -\frac{3}{4\pi^2}\ln\eta(e^{-2\pi S})$ and $K = 24$.

[5]The analysis of [24] was done in the context of toroidal compactification of heterotic string theory. We are using a generalization of this result.

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and

$$-\frac{\pi N}{S_0^2} + 4\pi - \frac{K}{2S_0} \simeq 0\,. \tag{4.12}$$

In order to complete the computation of S_{BH} we need to calculate the constant K.[6] Fortunately there is a simple expression for K by virtue of the fact that it appears as the coefficient of the non-holomorphic piece of $\phi(S,\bar{S})$ and hence is directly related to the holomorphic anomaly $\partial_S\partial_{\bar{S}}\phi(S,\bar{S})$ [43, 44]. This computation is carried out by mapping the heterotic string theory to the dual type IIA description. As is well known, heterotic string theory on T^4 is dual to type IIA string theory on K3 [56 – 60]. Under this duality the Narain lattice $\Lambda_{20,4}$ of the heterotic string theory gets mapped to the lattice of integer homology cycles of $K3$, and the components of the gauge fields take value in the real cohomology group of $K3$ [58]. Also the generator g of the $\mathbb{Z}_m$ symmetry of $\Lambda_{20,4}$, that was used in section 3 for the construction of the CHL model, gets mapped to an order m symmetry generator $\widetilde{g}$ of the conformal field theory describing type IIA string theory on $K3$ [51] with specific action on the elements of the homology and the cohomology group of $K3$. Since g preserves $(24-k)$ of the 24 directions of the Narain lattice associated with heterotic string compactification on T^4, $\widetilde{g}$ will preserve $(24-k)$ of the 24 basis vectors of the real cohomology group of $K3$. Now compactifying both sides on $\widetilde{S}^1\times S^1$ we get a duality between heterotic string theory on $T^4\times\widetilde{S}^1\times S^1$ and type IIA on $K3\times\widetilde{S}^1\times S^1$. Let us denote by h the generator of the order m shift along $\widetilde{S}^1$. Then the CHL model, obtained by modding out heterotic string theory on $T^4\times\widetilde{S}^1\times S^1$ by the $\mathbb{Z}_m$ symmetry group generated by $h\cdot g$, is dual to type IIA string theory on $K3\times\widetilde{S}^1\times S^1$, modded out by the $\mathbb{Z}_m$ group generated by $h\cdot\widetilde{g}$ [48 – 51]. We shall denote by $\mathcal{C}$ the conformal field theory associated with the six compact directions of the type IIA string theory after taking this quotient.

It is well known that the dilaton-axion field S of the heterotic string theory gets mapped to the complexified Kahler modulus of the two torus $\widetilde{S}^1\times S^1$ on the type IIA side [58]. Thus computation of $\partial_S\partial_{\bar{S}}\phi(S,\bar{S})$ requires computing the derivative of ϕ with respect to the Kahler modulus of $\widetilde{S}^1\times S^1$ and its complex conjugate in the type IIA description. The detailed procedure for this computation can be found in [43, 44]; here we just summarize the relevant result of these papers which lead to the value of K. Let us denote by ψ^4, ψ^5 the right-handed world-sheet fermions, and by $\bar{\psi}^4$, $\bar{\psi}^5$ the left-handed world-sheet fermions associated with the directions along the circles S^1 and $\widetilde{S}^1$ in the type IIA theory. We define:

$$\psi^\pm = \frac{1}{\sqrt{2}}(\psi^4 \pm i\psi^5)\,, \qquad \bar{\psi}^\pm = \frac{1}{\sqrt{2}}(\bar{\psi}^4 \pm i\bar{\psi}^5)\,. \tag{4.13}$$

In the Ramond-Ramond (RR) sector $\psi^\pm$ as well as $\bar{\psi}^\pm$ have zero modes. We denote them by $\psi_0^\pm$ and $\bar{\psi}_0^\pm$ respectively. They satisfy the usual anti-commutation relations

$$\{\psi_0^+, \psi_0^-\} = 1\,, \qquad \{\bar{\psi}_0^+, \bar{\psi}_0^-\} = 1\,, \tag{4.14}$$

[6]This calculation has been carried out earlier in [53] using direct analysis of one loop amplitudes in type II string theory.

with all other anti-commutators being zero. If we now define

$$C = \psi_0^+ \bar\psi_0^- , \qquad \bar C = \psi_0^- \bar\psi_0^+ , \tag{4.15}$$

then in the subspace of RR sector ground states they represent the action of the operators $\psi^+\bar\psi^-$ and $\psi^-\bar\psi^+$ which appear in the vertex operator of the Kahler class of $\widetilde S^1 \times S^1$ and its complex conjugate, — the fields S, $\bar S$ with respect to which we want to take derivatives of $\phi(S,\bar S)$. In terms of these operators the coefficient K is given by [43]

$$K = -\,\mathrm{Tr}_{RR}\left[(-1)^{F_L+F_R}\, C\, \bar C\right], \tag{4.16}$$

where the trace is to be taken over the RR sector ground states (with $L_0 = \bar L_0 = 0$) of the conformal field theory $\mathcal{C}$, and F_L and F_R denote the world-sheet fermion numbers in the left and the right-moving sector of this conformal field theory. In arriving at (4.16) we have used the fact that $\mathrm{Tr}((-1)^{F_L+F_R})$ vanishes in the conformal field theory $\mathcal{C}$, since the action of the fermion zero modes $\psi_0^\pm$ pairs states with equal and opposite $(-1)^{F_L+F_R}$ eigenvalues.

The states of the conformal field theory $\mathcal{C}$ include both untwisted sector states and twisted sector states. Of these the twisted sector states necessarily carry fractional unit of winding along $\widetilde S^1$. Hence the twisted RR states always have strictly positive L_0, $\bar L_0$ eigenvalues and cannot contribute to (4.16). The untwisted sector states are states associated with the original CFT with target space $K3 \times \widetilde S^1 \times S^1$ which are invariant under the $\mathbb{Z}_m$ symmetry generated by $h \cdot \tilde g$. These can be divided into two classes: those which are invariant separately under h and $\widetilde g$, and those which pick up equal and opposite non-trivial phases under the action of h and $\widetilde g$. The latter class, being not invariant under h, carries non-zero momentum along $\widetilde S^1$, and hence the RR sector states in this class have strictly positive L_0 and $\bar L_0$ eigenvalues. Thus they cannot contribute to the trace in (4.16), and we are left with states which are invariant separately under h and $\widetilde g$. Since the operators C and $\bar C$ appearing in (4.16) act on the Hilbert space of the CFT with target space $\widetilde S^1 \times S^1$, the contribution to K from these states may be factorized as

$$K = -\,\mathrm{Tr}_{RR}^{K3;\mathrm{inv}}\left[(-1)^{F_L+F_R}\right]\, \mathrm{Tr}_{RR}^{\widetilde S^1\times S^1}\left[(-1)^{F_L+F_R} C\bar C\right], \tag{4.17}$$

where in $\mathrm{Tr}_{RR}^{K3;inv}$ the trace is now taken over the $\widetilde g$ invariant RR sector ground states of the conformal field theory with target space $K3$, and in $\mathrm{Tr}_{RR}^{\widetilde S^1\times S^1}$ the trace is to be taken over the RR sector ground states of the CFT with target space $\widetilde S^1 \times S^1$. For the later CFT the requirement of vanishing L_0 and $\bar L_0$ forces the states to carry vanishing momenta along S^1 and $\widetilde S^1$ and hence they are automatically invariant under h.

There is a one to one map between the vector space of RR sector ground states in the CFT associated with K3 and the real cohomology group of $K3$. Under this map $(F_L + F_R)$ get mapped to the degree of the cohomology element. Since K3 has non-trivial cohomology of even degree only, $\mathrm{Tr}[(-1)^{F_L+F_R}]$ for K3 is equal to the dimension of the cohomology group of K3 which is 24. Here however we are interested in the RR sector ground states which are even under $\widetilde g$, and hence we should count the dimension of the cohomology group of K3 which is invariant under $\widetilde g$. Since this number is equal to $(24-k)$, we have

$$\mathrm{Tr}_{RR}^{K3;\mathrm{even}}\left[(-1)^{F_L+F_R}\right] = 24 - k\,. \tag{4.18}$$

In order to calculate the second factor appearing in (4.17), we note that the RR sector ground states associated with $\widetilde{S}^1 \times S^1$ consist of four states. Defining the vacuum state $|0\rangle$ to be annihilated by ψ_0^- and $\bar\psi_0^-$, and have $F_L = F_R = -\frac{1}{2}$, the states are

$$|0\rangle\,, \qquad \psi_0^+|0\rangle\,, \qquad \bar\psi_0^+|0\rangle\,, \qquad \psi_0^+\bar\psi_0^+|0\rangle\,, \tag{4.19}$$

with (F_L, F_R) values $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$ respectively. From the structure of C and $\bar C$ defined in (4.15) it is clear that only the state $\psi_0^+|0\rangle$ will contribute to the trace appearing in the second factor of (4.17). Since $C\bar C\psi_0^+|0\rangle = -\psi_0^+|0\rangle$, we get

$$\mathrm{Tr}_{RR}^{\widetilde{S}^1\times S^1}\left[(-1)^{F_L+F_R} C\bar C\right] = -1\,. \tag{4.20}$$

Substituting (4.18) and (4.20) into (4.17) we get

$$K = (24 - k)\,. \tag{4.21}$$

Eqs. (4.11), (4.12) now give

$$S_{BH} \simeq \frac{\pi N}{S_0} + 4\pi\, S_0 - \frac{1}{2}\,(24-k)\,\ln(2S_0)\,, \tag{4.22}$$

and

$$-\frac{\pi\, N}{S_0^2} + 4\pi - \frac{24-k}{2S_0} \simeq 0\,. \tag{4.23}$$

These agree with eqs.(3.13), (3.14) under the identification

$$\widetilde{S}_{\mathrm{stat}} = S_{BH}\,, \qquad \mu = \frac{\pi}{S_0}\,. \tag{4.24}$$

Thus we see that in this approximation the entropy S_{BH} of the black hole agrees with the statistical entropy $\widetilde{S}_{\mathrm{stat}}$ calculated following the procedure given in section 2 up to an overall constant.

Earlier we had estimated the error in $\widetilde{S}_{\mathrm{stat}}$ calculated from (3.13), (3.14) to be nonperturbative in $1/\sqrt{N}$. We shall now try to carry out a similar estimate of the error involved in eqs.(4.22), (4.23) so that we can determine up to what level the agreement between S_{BH} and $\widetilde{S}_{\mathrm{stat}}$ holds. First of all we should remember that there is an uncertainty involved in the original formulae (4.9), (4.10) since they have not been derived from first principles. In particular [24] used an argument based on duality symmetry to derive the effect of the holomorphic anomaly terms, and this does not fix the additive constant on the right hand side of (4.9). Thus there is an ambiguity in the overall additive constant in the expression for S_{BH}, and hence we cannot hope to compare the additive constants in $\widetilde{S}_{\mathrm{stat}}$ and S_{BH}. Assuming that the formulæ (4.9), (4.10) are correct up to this additive constant, we see that the error in the determination of S_{BH} lies essentially in the error in the determination of the function $g(S)$. As reviewed in appendix B, we can determine the form of the corrections to $g(S)$ by the requirement of S-duality invariance of the theory,[7] and typically corrections to (4.7) involve powers of $e^{-2\pi S}$. Since S_0 obtained by solving eq.(4.23) is of order $\sqrt{N}$, we see that the corrections to S_{BH} will involve powers of $e^{-\pi\sqrt{N}}$. Thus for large N the agreement between S_{BH} and $\widetilde{S}_{\rm stat}$ holds up to an undetermined additive constant, and terms which are non-perturbative in $1/\sqrt{N}$. In particular if we express $\widetilde{S}_{\rm stat}$ and S_{BH} in power series expansion in $1/\sqrt{N}$ by solving eqs.(3.13), (3.14) and (4.22), (4.23) respectively, then the results agree to all orders in $1/N$ including terms proportional to $\ln N$. This in turn implies similar agreement between $\ln d_N$ and $\ln d_N^{BH}$ defined in section 2.

5. Discussion

Given the agreement between the statistical entropy $\widetilde{S}_{\rm stat}$ and the black hole entropy S_{BH} up to non-perturbative terms, one might wonder if this correspondence also holds after we include non-perturbative terms. Unfortunately however even for toroidally compactified heterotic string theory $\widetilde{S}_{\rm stat}$ and S_{BH} differ once we include non-perturbative corrections [35, 45], and hence we expect that such disagreement will also be present for CHL compactifications. One could contemplate several reasons for this discrepancy:

1. First of all we should remember that the formulæ (4.9), (4.10) for the black hole entropy in the presence of non-holomorphic terms, as given in [24], have not been derived from first principles. Thus there could be further corrections to these formulæ which could modify the expression for S_{BH}.

2. It could also be that the proposal (2.1)–(2.4) for relating the black hole entropy to the degeneracy of elementary string states is not complete; and that the formulæ needs to be modified once non-perturbative effects are taken into account.

3. Besides supersymmetric completion of the curvature squared terms, the effective field theory contains infinite number of other higher derivative terms which are in principle equally important, and at present there is no understanding as to why these terms do not affect the expression for the entropy. It could happen that while these terms do not affect the perturbative corrections, their contribution becomes significant at the non-perturbative level.

4. Finally there is always a possibility that the relation between the black hole entropy and statistical entropy exists only as a power series expansion in inverse powers of various charges. In this case we do not expect any relation between the non-perturbative terms in the expressions for S_{BH} and $\widetilde{S}_{\rm stat}$.

At present we do not know which (if any) of these possibilities is correct. This issue clearly needs further investigation. We note however that if the fourth proposal is correct, namely the agreement between the black hole entropy and the statistical entropy holds only as a power series expansion in inverse powers of the charges, then the proposal (2.4) relating

[7] As will be discussed in appendix B, the S-duality group for these models is usually a subgroup of $SL(2,\mathbb{Z})$, and depends on the specific model we are analysing [49].

the black hole and the statistical entropy can be extended to more general models and more general states. The essential point is that in our analysis we have been restricted to compactifications of the form $K_5 \times S^1$ and to states carrying momentum and winding along S^1 in order to ensure that the degeneracy of the states depends only on the combination $N = nw$. If we consider more general $N = 4$ supersymmetric compactification (e.g. where the orbifold group is $\mathbb{Z}_m \times \mathbb{Z}_{m'}$ and acts on both circles instead of just one circle [51]) and/or more general states carrying arbitrary gauge charges $(\vec{P}_L, \vec{P}_R)$ associated with gauge fields arising out of the left and the right sectors of the world-sheet, then the role of the T-duality invariant combination nw is played by $N \equiv \frac{1}{2}(\vec{P}_R^2 - \vec{P}_L^2)$. However in this case the degeneracy $d(\vec{P}_L, \vec{P}_R)$ of such states could depend on $\vec{P}_L$ and $\vec{P}_R$ separately instead of being a function of the combination N only. To see how such dependence can arise, we can consider the class of models described in section 3 and consider a state that carries $\tilde{n}$ units of momentum along the circle $\widetilde{S}^1$ besides the charge quantum numbers n, w associated with the circle S^1. For such states we still have $N = nw$. However in this case the part of the wave-function associated with $\widetilde{S}^1$ picks up a phase $e^{2\pi i\tilde{n}/m}$ under the $\mathbb{Z}_m$ shift along $\widetilde{S}^1$, and we must compensate for it by introducing a factor of $e^{2\pi i\tilde{n}l/m}$ multiplying g^l in the projection operator (A.1) in order to ensure that the complete state is invariant under the $\mathbb{Z}_m$ transformation. This introduces a specific dependence of the partition function and hence of the degeneracy of states on $\tilde{n}$. If in addition the state carries some gauge charges associated with the lattice $\Lambda_{20,4}$, then the sum over momentum $\vec{s}$ in (3.9), (3.10) might run over a shifted lattice, which is equivalent to replacing the $\exp(-\mu\vec{s}^2/2)$ factor in (3.10) by $\exp[-\mu(\vec{s} + \vec{K})^2/2]$ for some fixed vector $\vec{K}$ that depends on the component of $(\vec{P}_L, \vec{P}_R)$ along $\Lambda_{20,4}$. However by following the analysis given in section 3 and appendix A one can see that the dependence on $(\vec{P}_L, \vec{P}_R)$ introduced by either of these effects is exponentially suppressed, and hence if we are interested in $\ln d(\vec{P}_L, \vec{P}_R)$ as a power series expansion in inverse powers of charges, the result depends only on the combination $N = (\vec{P}_R^2 - \vec{P}_L^2)/2$. Similar analysis can also be carried out for the $\mathbb{Z}_m \times \mathbb{Z}_{m'}$ orbifold models. This allows us to define $\widetilde{S}_{\rm stat}(N)$ through eqs. (2.1)–(2.3) within this approximation. On the other hand from the results of [24] it also follows that the black hole entropy will also be a function of the combination $N = \frac{1}{2}(\vec{P}_R^2 - \vec{P}_L^2)$. An analysis similar to the one described in this paper can then be used to show that the correspondence (2.4) between the statistical entropy and the black hole entropy continues to hold for these more general class of states and/or models.

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A. Estimating error in the computation of statistical entropy

In this appendix we shall estimate corrections to eq. (3.10) arising out of the fact that we needed to sum over g invariant states with g being the generator of a $\mathbb{Z}_m$ group; but instead we summed over all states and divided the result by a factor of m. The correct expression would involve inserting in the sum over states a projection operator

$$\frac{1}{m}\sum_{l=0}^{m-1} g^l \tag{A.1}$$

that projects onto g invariant states. The $l=0$ term in the above expression reproduces the right hand side of (3.10). Thus we need to estimate the contribution due to the $l\neq 0$ terms.

g acts non-trivially on the left-moving oscillators as well as the internal momentum carried by the state. We can choose appropriate linear combination of the left moving world-sheet scalar fields so that the annihilation operator $a_{\alpha n}$ $(1\leq\alpha\leq 24,\ 1\leq n<\infty)$ associated with the α-th scalar field picks up a $\mathbb{Z}_m$ phase $e^{2\pi i k_\alpha/m}$ $(0\leq k_\alpha\leq (m-1))$ under the g transformation. In that case the action of g on the oscillators is represented by the operator

$$\exp\left(\frac{2\pi i}{m}\sum_{\alpha=1}^{24}\sum_{n=1}^{\infty} k_\alpha\, a^\dagger_{\alpha n} a_{\alpha n}\right). \tag{A.2}$$

Let $\widehat{g}$ denote the action of g on the internal momentum. Then the $l>0$ terms in the sum in (A.1) have the form

$$\frac{1}{m}\sum_{l=1}^{m-1}\exp\left(\frac{2\pi i l}{m}\sum_{\alpha=1}^{24}\sum_{n=1}^{\infty} k_\alpha\, a^\dagger_{\alpha n} a_{\alpha n}\right)\widehat{g}^l\,. \tag{A.3}$$

Inserting this into the trace over BPS states weighted by $e^{-\mu(N_L-1+\vec{s}^2/2)}$ we see first of all that unless $\widehat{g}^l\vec{s}=\vec{s}$ the lattice vector $\vec{s}$ does not contribute to the trace. Thus the momentum sum receives contribution only from a sublattice Λ_l of Λ which is invariant under $\widehat{g}^l$. Since Λ is transverse to the directions in $\Lambda_{20,4}$ which are left invariant under $\widehat{g}$, $\widehat{g}$ does not preserve any direction of the lattice Λ. Thus for $l=1$ the sublattice Λ_l consists of the single point $\vec{s}=0$. But if m is not prime, then $\widehat{g}^l$ can be of order $<m$ for some l and Λ_l could be non-trivial. On the other hand, contribution to the l-th term in $e^{\mathcal{F}(\mu)}$ from the oscillators $(a_{\alpha n}, a^\dagger_{\alpha n})$ is given by

$$\mathrm{Tr}_{\alpha,n}\left(e^{-\mu n a^\dagger_{\alpha n}\, a_{\alpha n}+2\pi i l k_\alpha a^\dagger_{\alpha n}\, a_{\alpha n}/m}\right)=\frac{1}{1-e^{2\pi i l k_\alpha/m}e^{-n\mu}}\,, \tag{A.4}$$

where $\mathrm{Tr}_{\alpha,n}$ denotes trace over all states created by (multiple) application of the oscillator $a^\dagger_{\alpha n}$ on the vacuum. Thus the net additional contribution to $e^{\mathcal{F}(\mu)}$ from the $l\neq 0$ terms is given by:

$$\frac{1}{m}\sum_{l=1}^{m-1}\prod_{\alpha=1}^{24}\prod_{n=1}^{\infty}\frac{1}{1-e^{2\pi i l k_\alpha/m}q^n}\sum_{\vec{s}\in\Lambda_l}e^{-\mu\vec{s}^2/2}\,,\qquad q\equiv e^{-\mu}\,. \tag{A.5}$$

We need to compare this with the $l=0$ term. First of all we see that contribution from the momentum sum, running over a sublattice of Λ, is suppressed by a power of μ that depends on the difference in the dimension of the original lattice Λ and the sublattice Λ_l. But more importantly, the contribution from the α-th oscillator is suppressed by a factor of

$$\prod_{n=1}^{\infty}\frac{1}{1-e^{2\pi i l k_\alpha/m}q^n}\Big/\prod_{n=1}^{\infty}\frac{1}{1-q^n}\,. \tag{A.6}$$

For small μ this ratio is suppressed by a factor of [4]

$$\left(\frac{4\pi}{\mu}\sin\frac{\pi k_\alpha l}{m}\right)^{1/2}\exp\left[-\frac{\pi^2}{\mu}\frac{k_\alpha l}{m}\left(1-\frac{k_\alpha l}{m}\right)\right], \tag{A.7}$$

for $0 < k_\alpha l/m < 1$. This shows that the correction to $\mathcal{F}(\mu)$ given in (3.10) involves powers of $e^{-\pi^2/\mu}$. The argument given below eq.(3.15) now leads to the conclusion that the error in $\widetilde{S}_{\rm stat}$ computed from (3.13), (3.14) involves powers of $e^{-\pi\sqrt{N}}$.

B. Estimating error in the computation of black hole entropy

In this appendix we shall analyze the function $g(S)$ and estimate the error in the expression (4.7) for this function.[8] The basic tool used in this analysis will be S-duality invariance. The S-duality group of the CHL models of the type considered in this paper may be found by identifying it as the T-duality group in the dual type IIA description [49]. It is a subgroup of the SL$(2,\mathbb{Z})$ group that commutes with the orbifold group and acts on S as

$$S \to -i\frac{iaS+b}{icS+d} \qquad ad-bc=1\,, \qquad a,b,c,d\in\mathbb{Z}\,, \tag{B.1}$$

together with some additional restriction on a, b, c, d. For the $\mathbb{Z}_m$ orbifold models described in section 3 this additional restriction takes the form [49]:[9]

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 \bmod m\\ 0 \bmod m\end{pmatrix}. \tag{B.2}$$

The resulting group is known as $\Gamma_0(m)$ [61].[10] It was shown in [24] that in order that the supergravity effective action is invariant under S-duality after addition of the term (4.5) to the action, the combination

$$h(S,\bar S) \equiv \partial_S\left(g(S) - \frac{K}{64\pi^2}\,\ln(S+\bar S)\right) \tag{B.3}$$

must transform under a duality transformation as:

$$h\left(-i\frac{iaS+b}{icS+d}, i\frac{-ia\bar S+b}{-ic\bar S+d}\right) = (icS+d)^2 h(S,\bar S)\,. \tag{B.4}$$

Using the known modular transformation laws of $\eta\left(e^{-2\pi S}\right)$ and $S+\bar S$ it then follows that the holomorphic combination

$$\widehat{h}(S) \equiv h(S,\bar S) + \frac{K}{64\pi^2}\,\partial_S\left[\ln\eta^2\left(e^{-2\pi S}\right)+\ln(S+\bar S)\right] = \partial_S g(S) + \frac{K}{32\pi^2}\,\partial_S\ln\eta\left(e^{-2\pi S}\right) \tag{B.5}$$

[8]For some orbifold models similar analysis has been carried out in [53].

[9]I wish to thank P. Aspinwall for discussion on this point.

[10]We identify elements of $\Gamma_0(m)$ related to each other by $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \to \begin{pmatrix} -a & -b \\ -c & -d\end{pmatrix}$ since (B.1) is unchanged under this transformation.

transforms as a modular form of weight two:

$$\widehat{h}\left(-i\frac{iaS+b}{icS+d}\right) = (icS+d)^2\widehat{h}(S)\,. \tag{B.6}$$

Furthermore, from (4.7) and (B.5) it follows that for large S,

$$\widehat{h}(S) \simeq \frac{1}{16\pi}\left(1-\frac{K}{24}\right). \tag{B.7}$$

For toroidal compactification of heterotic string theory the S-duality group is $\mathrm{SL}(2,\mathbb{Z})$ which has no modular form of weight two. Thus $\widehat{h}(S)$ must vanish. This is consistent with eq. (B.7) since $K = 24$ for toroidal compactification. In general however K given in (4.21) is less than 24, and hence $\widehat{h}(S)$ must be non-trivial. Fortunately $\Gamma_0(m)$ for $m \geq 2$ does have non-trivial modular forms of weight two [61]. If m is prime then this modular form is unique and explicit expression for this modular form is given by [61]:

$$E(iS;m) = mG_2^*(imS) - G_2^*(iS)\,, \tag{B.8}$$

where

$$G_2^*(iS) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n_1,n_2\geq 1} n_1 e^{-2\pi n_1 n_2 S} = -4\pi\partial_S \ln\eta\left(e^{-2\pi S}\right). \tag{B.9}$$

$E(iS;m)$ is normalized such that for $S\to\infty$, $E(iS;m)\to(m-1)\pi^2/3$. Thus eq.(B.7) gives

$$\widehat{h}(S) = \frac{3}{16\pi^3(m-1)}\left(1-\frac{K}{24}\right)E(iS;m)\,. \tag{B.10}$$

$g(S)$ can now be determined from (B.5) up to an overall additive constant.

If m is not prime then $\widehat{h}(S)$ is not determined uniquely by this argument. Nevertheless since modular forms of $\Gamma_0(m)$ have series expansion in powers of $e^{-2\pi S}$, we see that the correction to (B.7) is of order $e^{-2\pi S}$. This in turn implies that the corrections to (4.7) involve powers of $e^{-2\pi S}$ and hence are non-perturbative in $1/S$. We can now invoke the analysis below eq. (4.24) to conclude that the error in S_{BH} computed from eqs. (4.22), (4.23) involves powers of $e^{-\pi\sqrt{N}}$.

References

[1] G. 't Hooft, *The black hole interpretation of string theory*, *Nucl. Phys.* **B 335** (1990) 138.

[2] L. Susskind, *Some speculations about black hole entropy in string theory*, hep-th/9309145.

[3] L. Susskind and J. Uglum, *Black hole entropy in canonical quantum gravity and superstring theory*, *Phys. Rev.* **D 50** (1994) 2700 [hep-th/9401070].

[4] J.G. Russo and L. Susskind, *Asymptotic level density in heterotic string theory and rotating black holes*, *Nucl. Phys.* **B 437** (1995) 611 [hep-th/9405117].

[5] G.T. Horowitz and J. Polchinski, *A correspondence principle for black holes and strings*, *Phys. Rev.* **D 55** (1997) 6189 [hep-th/9612146].

JHEP07(2005)063

[6] A. Dabholkar and J.A. Harvey, *Nonrenormalization of the superstring tension*, *Phys. Rev. Lett.* **63** (1989) 478.

[7] A. Dabholkar, G.W. Gibbons, J.A. Harvey and F. Ruiz Ruiz, *Superstrings and solitons*, *Nucl. Phys.* **B 340** (1990) 33.

[8] A. Dabholkar, J.P. Gauntlett, J.A. Harvey and D. Waldram, *Strings as solitons & black holes as strings*, *Nucl. Phys.* **B 474** (1996) 85 [hep-th/9511053].

[9] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A 10** (1995) 2081 [hep-th/9504147].

[10] A.W. Peet, *Entropy and supersymmetry of D-dimensional extremal electric black holes versus string states*, *Nucl. Phys.* **B 456** (1995) 732 [hep-th/9506200].

[11] A. Sen, *Black holes and elementary string states in $N = 2$ supersymmetric string theories*, *JHEP* **02** (1998) 011 [hep-th/9712150].

[12] J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, hep-th/9602052.

[13] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[14] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[15] A. Strominger, *Macroscopic entropy of $N = 2$ extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111].

[16] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D 54** (1996) 1514 [hep-th/9602136].

[17] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [hep-th/9711053].

[18] B. de Wit, *$N = 2$ electric-magnetic duality in a chiral background*, *Nucl. Phys.* **49** *(Proc. Suppl.)* (1996) 191 [hep-th/9602060].

[19] B. de Wit, *$N = 2$ symplectic reparametrizations in a chiral background*, *Fortschr. Phys.* **44** (1996) 529 [hep-th/9603191].

[20] B. Zwiebach, *Curvature squared terms and string theories*, *Phys. Lett.* **B 156** (1985) 315.

[21] J.A. Harvey and G.W. Moore, *Fivebrane instantons and $\mathbb{R}^2$ couplings in $N = 4$ string theory*, *Phys. Rev.* **D 57** (1998) 2323 [hep-th/9610237].

[22] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [hep-th/9812082].

[23] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, *Fortschr. Phys.* **48** (2000) 49 [hep-th/9904005].

[24] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, *Nucl. Phys.* **B 567** (2000) 87 [hep-th/9906094].

[25] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Area law corrections from state counting and supergravity*, *Class. and Quant. Grav.* **17** (2000) 1007 [hep-th/9910179].

JHEP07(2005)063

[26] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortschr. Phys.* **49** (2001) 3 [hep-th/0007195].

[27] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $N = 2$ supergravity with $\mathbb{R}^2$ interactions*, *JHEP* **12** (2000) 019 [hep-th/0009234].

[28] G.L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Examples of stationary BPS solutions in $N = 2$ supergravity theories with $\mathbb{R}^2$-interactions*, *Fortschr. Phys.* **49** (2001) 557 [hep-th/0012232].

[29] R.M. Wald, *Black hole entropy in the Noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [gr-qc/9307038].

[30] T. Jacobson, G. Kang and R.C. Myers, *On black hole entropy*, *Phys. Rev.* **D 49** (1994) 6587 [gr-qc/9312023].

[31] V. Iyer and R.M. Wald, *Some properties of noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D 50** (1994) 846 [gr-qc/9403028].

[32] T. Jacobson, G. Kang and R.C. Myers, *Black hole entropy in higher curvature gravity*, gr-qc/9502009.

[33] A. Dabholkar, *Exact counting of black hole microstates*, hep-th/0409148.

[34] A. Dabholkar, R. Kallosh and A. Maloney, *A stringy cloak for a classical singularity*, *JHEP* **12** (2004) 059 [hep-th/0410076].

[35] A. Sen, *How does a fundamental string stretch its horizon?*, *JHEP* **05** (2005) 059 [hep-th/0411255].

[36] V. Hubeny, A. Maloney and M. Rangamani, *String-corrected black holes*, *JHEP* **05** (2005) 035 [hep-th/0411272].

[37] D. Bak, S. Kim and S.J. Rey, *Exactly soluble BPS black holes in higher curvature $N = 2$ supergravity*, hep-th/0501014.

[38] S. Giusto and S.D. Mathur, *Fuzzball geometries and higher derivative corrections for extremal holes*, hep-th/0412133.

[39] S.D. Mathur, *The fuzzball proposal for black holes: an elementary review*, hep-th/0502050.

[40] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[41] E.P. Verlinde, *Attractors and the holomorphic anomaly*, hep-th/0412139.

[42] A. Dabholkar, talk at ISM04 meeting at Khajuraho, India, Dec. 2004.

[43] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [hep-th/9302103].

[44] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[45] G.L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287].

[46] S. Chaudhuri, G. Hockney and J.D. Lykken, *Maximally supersymmetric string theories in $D < 10$*, *Phys. Rev. Lett.* **75** (1995) 2264 [hep-th/9505054].

JHEP07(2005)063

[47] S. Chaudhuri and J. Polchinski, *Moduli space of chl strings*, *Phys. Rev.* **D 52** (1995) 7168 [hep-th/9506048].

[48] J.H. Schwarz and A. Sen, *Type IIA dual of the six-dimensional CHL compactification*, *Phys. Lett.* **B 357** (1995) 323 [hep-th/9507027].

[49] C. Vafa and E. Witten, *Dual string pairs with $N = 1$ and $N = 2$ supersymmetry in four dimensions*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 225 [hep-th/9507050].

[50] S. Chaudhuri and D.A. Lowe, *Type IIA heterotic duals with maximal supersymmetry*, *Nucl. Phys.* **B 459** (1996) 113 [hep-th/9508144].

[51] P.S. Aspinwall, *Some relationships between dualities in string theory*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 30 [hep-th/9508154].

[52] A. Dabholkar, F. Denef, G. Moore and B. Piolone, to appear.

[53] A. Gregori et al., $\mathbb{R}^2$ *corrections and non-perturbative dualities of $N = 4$ string ground states*, *Nucl. Phys.* **B 510** (1998) 423 [hep-th/9708062].

[54] K.S. Narain, *New heterotic string theories in uncompactified dimensions ¡ 10*, *Phys. Lett.* **B 169** (1986) 41.

[55] K.S. Narain, M.H. Sarmadi and E. Witten, *A note on toroidal compactification of heterotic string theory*, *Nucl. Phys.* **B 279** (1987) 369.

[56] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B 438** (1995) 109 [hep-th/9410167].

[57] M.J. Duff, *Strong/weak coupling duality from the dual string*, *Nucl. Phys.* **B 442** (1995) 47 [hep-th/9501030].

[58] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124].

[59] A. Sen, *String string duality conjecture in six-dimensions and charged solitonic strings*, *Nucl. Phys.* **B 450** (1995) 103 [hep-th/9504027].

[60] J.A. Harvey and A. Strominger, *The heterotic string is a soliton*, *Nucl. Phys.* **B 449** (1995) 535 [hep-th/9504047].

[61] B. Schoeneberg, *Elliptic modular functions*, Springer Verlag, 1974.